

# Back to Basics: Sampling, Fourier Transform and Fast Fourier Transform (FFT)

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## Abstract

It can be easily argued that the most important topic of signal processing is the sampling of the continuous time signals and the consequences of this operation. An equally important topic is the Fourier transform and its conceptual and operational understanding. Typically, students are introduced to these topics as early as possible to establish a solid foundation for the forthcoming topics. In spite of these efforts, the approximate evaluation of the Fourier integral over a uniform grid via the Fast Fourier Transform (FFT) remains as a “tricky business” for the majority of students. This problem essentially stems from the separation of the continuous time Fourier transform related topics and FFT related topics in the classical textbooks such as [Oppenheim, Schaffer, Buck “Discrete-time Signal Processing”, Prentice Hall, 2009]. It is well known by the readers of earlier classical texts such as [Papoulis, “Signal Analysis”, 1977] that the connection between these topic is through the Poisson Summation formula, which is a topic at-large in almost all modern textbooks. This note aims to illustrate the procedure for retrieving the Fourier domain samples of a continuous time function and the nature of approximation when the samples are retrieved through an FFT operation.

## 1 Sampling Theorem and Fourier Series

We would like to underline the fact that the sampling operation and Fourier series expansion is essentially the same mathematical operation. To do that, we assume that  $f(t)$  is a continuous time function and the forward and inverse Fourier transform relations related to this function can be expressed as follows:

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft} dt, \quad f(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft} df.$$

A function is said to be limited, if its support is finite. It should be clear that a function can not be both time and band limited at the same time. Yet, it can be unlimited in both domains, the function  $f(t) = \exp(-\pi t^2)$  is an example for such functions. We would like to emphasize that the time and Fourier domain representations are duals of each other and several interesting relations are very simply derived by the utilization of this duality. We would like to suggest to the reader not to attribute more importance to one of these domains and treat both domains as alternative representations of the same function or operator.

We know that a time-limited function can be expanded into Fourier series by periodically replicating this function with the period  $T$ . More explicitly, if  $\tilde{f}(t)$  be the periodic replication of the time-limited function  $f(t)$ , i.e.  $\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t - nT)$ , then its Fourier series representation can be written as

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t - nT) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad (1)$$

where  $a_k = 1/T \times F(k/T)$ , that is  $a_k$  is the samples of Fourier domain representation of  $f(t)$  with some scaling.

Let's assume that  $f(t)$  is limited to  $T_{\text{lim}}$ , that is  $f(t) = 0$  for  $|t| > T_{\text{lim}}$  and the replication period  $T$  to form  $\tilde{f}(t)$  satisfies  $T > 2T_{\text{lim}}$ . Hence, from a single period of  $f(t)$ , we can exactly retrieve the time-limited function  $f(t)$  back. We would like to describe this important situation a little more succinctly as follows:

**Fact 1 (Fourier Series)** *A function limited in its own domain (which can be either time or Fourier domain) can be written as a series expansion whose expansion coefficients are the uniformly spaced, sufficiently dense samples of the function in the other domain, that is the domain in which the function is not limited.*

We would like to note that the fact given above treats both time and Fourier domains on equal basis. Hence, a function limited to  $F_{\text{lim}}$ , can be written as

$$\tilde{F}(f) = \sum_{n=-\infty}^{\infty} F(f - n\Delta F) = \sum_{k=-\infty}^{\infty} a_{-k} e^{j\frac{2\pi}{\Delta F}kf} = \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi}{\Delta F}kf} \quad (2)$$

where  $a_k = 1/\Delta F \times f(k/\Delta F)$ , that is  $a_k$  sequence is the samples of the function taken in the time-domain, i.e. the other domain, with some scaling. From Fact 1, we know that provided that the condition non-overlapping replicas is satisfied, that is  $\Delta F > 2F_{\text{lim}}$ ; we have

$$\tilde{F}(f) = F(f) = \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi}{\Delta F}kf}, \quad |f| < \frac{\Delta F}{2}; \quad \text{provided that } \Delta F > 2F_{\text{lim}}. \quad (3)$$

The same relation can also be written as follows, considering one more time that  $F(f)$  is limited to  $F_{\text{lim}}$ :

$$F(f) = \begin{cases} \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi}{\Delta F}kf}, & |f| < \frac{\Delta F}{2} \\ 0, & \text{Other} \end{cases}; \quad \text{provided that } \Delta F > 2F_{\text{lim}}. \quad (4)$$

Before the application of the inverse Fourier transform to both sides of (4), we recall that

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \iff \begin{cases} 1, & |f| < 1/2 \\ 0, & \text{Other} \end{cases}$$

and generalize this relation, via the scaling property,  $|a|f(at) \iff F(f/a)$  to

$$\Delta F \text{sinc}(\Delta f t) \iff \begin{cases} 1, & |f| < \Delta F/2 \\ 0, & \text{Other} \end{cases}.$$

and then apply the shift property,  $f(t - t_0) \iff F(f)e^{-j2\pi t_0 f}$ , to get

$$\Delta F \text{sinc}\left(\Delta f\left(t - \frac{k}{\Delta F}\right)\right) \iff \begin{cases} e^{-j\frac{2\pi}{\Delta F}kf}, & |f| < \Delta F/2 \\ 0, & \text{Other} \end{cases}. \quad (5)$$

The application of inverse Fourier transform to both sides of (4) can be easily evaluated with the Fourier transform pair given in (5):

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \Delta F \text{sinc}\left(\Delta f\left(t - \frac{k}{\Delta F}\right)\right), \quad \text{provided that } \Delta F > 2F_{\text{lim}}. \quad (6)$$

where  $a_k = 1/\Delta F \times f(k/\Delta F)$ . Denoting  $1/\Delta F$  with  $T_s$ , i.e. the sampling period, the equation (6) can be written as

**Sampling Theorem:**  $f(t) = \sum_{k=-\infty}^{\infty} f(kT_s) \text{sinc}\left(\frac{t}{T_s} - k\right)$ , provided that  $\frac{1}{T_s} = \Delta F > 2F_{\text{lim}}$ . (7)

The final equation is nothing but the sampling theorem for the bandlimited signals.

**Fact 2 (Sampling Theorem)** *A function limited either in time or Fourier domain can be expressed in the the other domain, the domain that the function is not limited, as a series expansion whose expansion coefficients are uniformly spaced, sufficiently dense samples of the function in the other domain.*

The main results of this section on a bandlimited function  $F(f)$  satisfying the condition  $F(f) = 0, |f| > F_{\text{lim}}$  are given by equations (3) and (7) and can be summarized as follows:

**Fourier Series:** 
$$F(f) = \frac{1}{\Delta f} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\Delta f}\right) e^{-j\frac{2\pi}{\Delta f}kf}, \quad |f| < \frac{\Delta F}{2}; \quad \text{provided that } \Delta F > 2F_{\text{lim}},$$

**Sampling Theorem:** 
$$f(t) = \sum_{k=-\infty}^{\infty} f(kT_s)\text{sinc}\left(\frac{t}{T_s} - k\right), \quad \text{provided that } \frac{1}{T_s} = \Delta F > 2F_{\text{lim}}. \quad (8)$$

**Fact 3 (Series Expansions for Limited Functions)** *If a function is limited in either time or Fourier domain, it is possible to express the time and Fourier domain representations of this function as an infinite series. The expansion using its own domain samples is called the sampling theorem, the expansion using the other domain samples is called the Fourier series. The expansion coefficients in both cases are the samples of function in the domain that the function is not limited.*

## 2 Computation of Spectrum Samples by DFT

We would like to compute the spectrum of the function  $f(t)$ . Typically, it is assumed that  $f(t)$  is a band-limited function and it is sampled with a sufficiently high rate to avoid aliasing. Ideally, the collected samples should be processed with the Discrete Time Fourier Transform (DTFT). A single period of DTFT spectrum is identical to the spectrum of the continuous function, apart from some scaling. Unfortunately, it is not possible to realize this operation in practice. This is due to two reasons. The number of samples to be processed by DTFT is infinitely many, since the function  $f(t)$  is band-limited and it is not possible to process infinitely many samples. The other reason is the inherent to the DTFT definition. DTFT is a mapping from discrete time to  $[0, 2\pi)$  continuum and it is defined through an integral relation whose numerical evaluation is not desirable in many cases. In practice, sufficiently large number of samples of  $f(t)$  is processed with DFT and with a rough interpretation, the DFT spectrum is interpreted as the samples of  $F(f)$ , apart from some scaling. This section aims to illustrate the details of this procedure and discusses the error associated with it.

We illustrate the procedure with the following function

$$f(t) = e^{-\frac{\pi}{2}t^2} + 2e^{-\frac{\pi}{2}(t-1)^2} + 2e^{-\frac{\pi}{2}(t+1)^2} + 2e^{-\frac{\pi}{2}(t-2.5)^2} + 2e^{-\frac{\pi}{2}(t+2.5)^2}, \quad (9)$$

whose Fourier transform can be explicitly expressed as

$$F(f) = \sqrt{2}e^{-2\pi f^2} (1 + 4\cos(2\pi f) + 4\cos(5\pi f)). \quad (10)$$

It should be noted that both  $f(t)$  and  $F(f)$  are real valued, even functions. Since both functions are not complex valued, we can easily present their sketch as shown in Figure 1. Our goal is to illustrate the application of the DFT based procedure to approximately retrieve the samples of the function  $F(f)$ .

Please note that both  $f(t)$  and  $F(f)$  are not strictly limited functions, but as shown in Figure 1, both functions rapidly decay for large enough  $t$  or  $f$  values. We may say that the time support of the

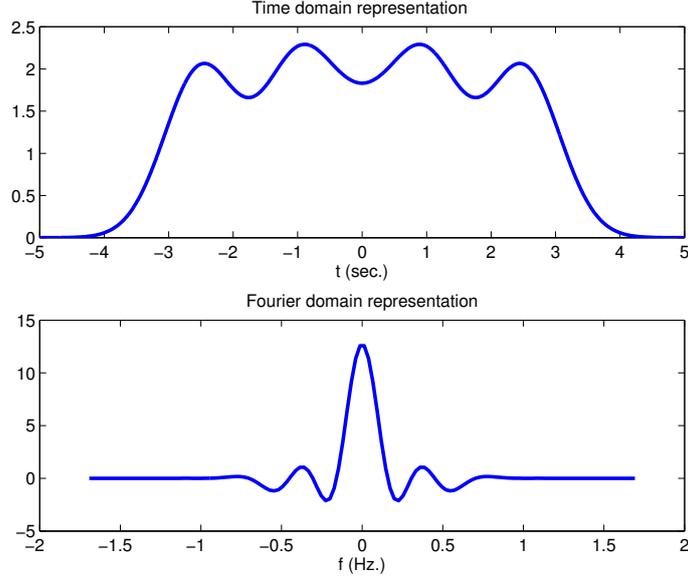


Figure 1: Time and Fourier domain representations of the function  $f(t)$

function is practically limited to  $[-5, 5]$  seconds and the frequency support is limited to  $[-1, 1]$  Hz. We can argue that this function lies in time-frequency plane with the area of 10 seconds  $\times$  2 Hertz = 20 (unitless). The conventional signal processing wisdom, which can be further justified via the prolate spheroidal functions, states that this function lies in a region with 20 degrees of freedom and it should be possible to express the function which is defined in a continuum either in time or frequency domain with approximately 20 samples.

We illustrate the steps for the calculation of spectrum samples through a running example. We present different cases to illustrate different situations.

**Case 1 (Aliasing in Frequency):** Figure 2a presents the functions  $f(t)$  and  $F(f)$  defined earlier. Figure 2b shows the time and frequency domain representations of the function when  $f(t)$  is sampled with  $T_s = 1$  Hz,

$$f[n] = f(nT_s). \quad (11)$$

As noted before, the sampling process in one domain leads to a periodic extension in the other domain. We denote the periodically extended signal as  $\tilde{F}_{f_s}(f)$  ( $f_s = 1/T_s$ ) as follows

$$\tilde{F}_{f_s}(f) = f_s \sum_{k=-\infty}^{\infty} F(f - kf_s). \quad (12)$$

It can be noted from the frequency domain picture of Figure 2b that a single period of  $\tilde{F}_{f_s}(f)$  does not resemble  $F(f)$  at all. This is essentially due to the repetition of the frequency domain signal with a period smaller than its (practical) support of 2 Hz. In the case of frequency aliasing, which is the case shown here, there is highly any need to further proceed with the calculation of spectrum samples. Yet, we proceed with the discussions to illustrate the process.

In the second step, shown in Figure 2c, the periodic signal  $\tilde{F}_{f_s}(f)$  is sampled with the rate of  $N_{\text{point}}$  samples per period. For the case illustrated here,  $N_{\text{point}} = 10$  and this corresponds to

$$f_p = f_s/N_{\text{point}} = \frac{1}{10} \text{ samples per Hertz} \quad (13)$$

for this case and we have

$$\tilde{F}_{f_s}[k] = \tilde{F}_{f_s}(kf_p). \quad (14)$$

As frequency sampling corresponds to periodic extension in time, the periodically extended time domain signal can be written as

$$\tilde{f}_{T_p}[n] = T_p \sum_{k=-\infty}^{\infty} f[n - kT_p f_s] = T_p \sum_{k=-\infty}^{\infty} f[n - kN_{\text{point}}]. \quad (15)$$

where in the last relation  $T_p = 1/f_p$  is the period of the extension with the units of seconds, and  $T_p f_s$  is the same quantity with the units of samples, which is collected at the rate of  $f_s$  Hertz.

The plots in Figure 2d shows a single period of periodic signals resulting from time and frequency domain sampling. We reiterate that the signals shown in Figure 2d are related with Fourier Transformation, furthermore we may expect that the periodic discrete time sequences shown in this figure to be the discrete Fourier series expansion of each other. To justify this expectation, we explicitly calculate the  $N_{\text{point}}$  DFT of the time-domain signal and compare the result with the frequency domain result shown in Figure 2d.

For the evaluation of  $N_{\text{point}}$  DFT of  $\tilde{f}_{T_p}[n]$ , we focus on the period  $\tilde{f}_{T_p}[n]$  from  $n = 0$  to  $n = N_{\text{point}} - 1$ .

$$\begin{aligned}
\sum_{n=0}^{N_{\text{point}}-1} \tilde{f}_{T_p}[n] e^{-j \frac{2\pi}{N_{\text{point}}} kn} &= \sum_{n=0}^{N_{\text{point}}-1} \left( T_p \sum_{k'=-\infty}^{\infty} f[n - k' N_{\text{point}}] \right) e^{-j \frac{2\pi}{N_{\text{point}}} kn} \\
&= T_p \sum_{k'=-\infty}^{\infty} \sum_{n=0}^{N_{\text{point}}-1} f[n - k' N_{\text{point}}] e^{-j \frac{2\pi}{N_{\text{point}}} kn} \underbrace{e^{j \frac{2\pi}{N_{\text{point}}} k k' N_{\text{point}}}}_1 \\
&= T_p \sum_{k'=-\infty}^{\infty} \sum_{n=0}^{N_{\text{point}}-1} f[n - k' N_{\text{point}}] e^{-j \frac{2\pi}{N_{\text{point}}} k(n - k' N_{\text{point}})} \\
&= T_p \sum_{n=-\infty}^{\infty} f[n] e^{-j \frac{2\pi}{N_{\text{point}}} kn} \\
&= T_p \sum_{n=-\infty}^{\infty} f[n] e^{-j 2\pi f' n}, \quad (f' = k/N_{\text{point}}) \\
&= T_p \sum_{n=-\infty}^{\infty} f(nT_s) e^{-j \frac{2\pi}{f_s} (f_s f') n}, \quad (f' = k/N_{\text{point}}) \\
&= T_p \sum_{n=-\infty}^{\infty} f(nT_s) e^{-j \frac{2\pi}{f_s} f'' n}, \quad (f'' = f_s k/N_{\text{point}}) \\
&= \underbrace{T_p f_s}_{N_{\text{point}}} T_s \sum_{n=-\infty}^{\infty} f(nT_s) e^{-j \frac{2\pi}{f_s} f'' n}, \quad (f'' = f_s k/N_{\text{point}}) \\
&\stackrel{(8), F.S.}{=} N_{\text{point}} T_s \sum_{k=-\infty}^{\infty} F(f'' - k f_s), \quad (f'' = f_s k/N_{\text{point}}) \\
&\stackrel{(12)}{=} N_{\text{point}} T_s \tilde{F}_{f_s}(f''), \quad (f'' = f_s k/N_{\text{point}} \stackrel{(13)}{=} f_p k) \\
&= N_{\text{point}} T_s \tilde{F}_{f_s}(f_p k) \\
&\stackrel{(14)}{=} N_{\text{point}} T_s \tilde{F}_{f_s}[k] \tag{16}
\end{aligned}$$

The final relation shows that samples of the periodic functions are the  $N$ -point DFT of each other, apart from a spacing by  $T_p$ ,

$$\frac{f_s}{N_{\text{point}}} \tilde{f}_{T_p}[n] \stackrel{N_{\text{point}} \text{ DFT}}{\iff} \tilde{F}_{f_s}[k]. \tag{17}$$

**Case 2 (Aliasing in Time):** We revisit the same problem. This time the time-domain signal is sampled with  $f_s = 2$  Hz. The periodically replicated spectrum is practically free of aliasing effects as can be seen from the Fourier domain plot of Figure3b. We would like to note that the sampling process shown in Figure3a results in a 21 non-zero samples in the interval  $[-5, 5]$ .

In the second step, the periodically replicated spectrum is sampled at  $N_{\text{point}} = 10$  points as shown in Figure3c. This corresponds to a frequency domain sampling of  $f_p = f_s/N_{\text{point}} = 1/5$  samples per Hertz. The frequency domain sampling results in the periodic extension of the signal with the period of  $T_p = 1/f_p = 5$  seconds. Since, the signal is already in discrete time, this corresponds to the

replication of 21 element vector given in the top part of Figure3c with the period of  $5f_s = 10$  samples. This causes an aliasing in the time-domain.

In the third step, a single period of discrete time sequences with the period  $N_{\text{point}} = 10$  samples is shown. As explained above, the relation between periodic sequences in time and frequency domain is simply discrete time Fourier series or equivalently the DFT.

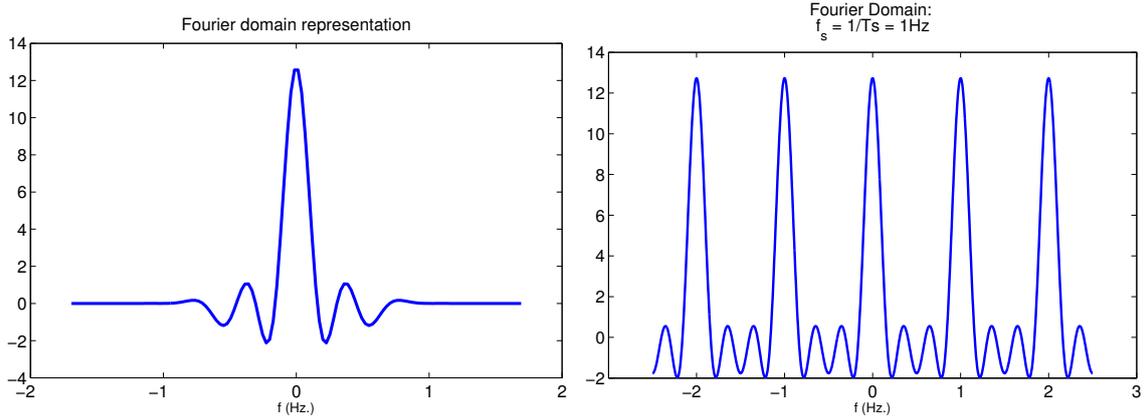
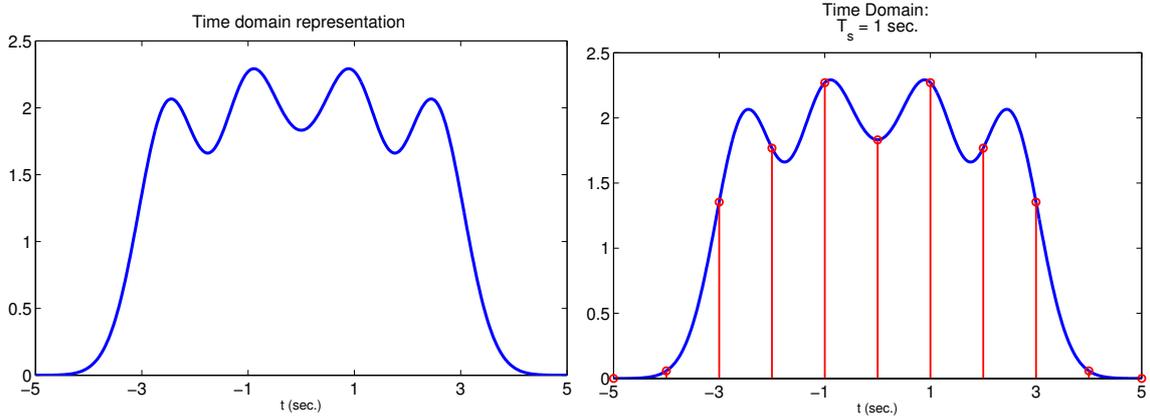
**Case 3 (no evident aliasing effect):** For this case, we sample the time-domain signal with  $f_s = 2$  Hz as in Case 2. The resulting signal is a vector 21 non-zero entries and there is evident no aliasing in the frequency domain, as shown in Figure4b. It should be noted both of these remarks stating 21 non-zero entries for time-domain representation and no evident aliasing in frequency, are only approximately true, since we know that the original function only asymptotically reaches zero in both frequency and time. But, for all practical purposes, these remarks can be considered as correct.

In the second step, the periodically replicated spectrum is sampled at  $N_{\text{point}} = 20$  points as shown in Figure4c. This corresponds to a frequency domain sampling of  $f_p = f_s/N_{\text{point}} = 1/10$  samples per Hertz. The frequency domain sampling results in the periodic extension of the signal with the period of  $T_p = 1/f_p = 10$  seconds. Since, the signal is already in discrete time, this corresponds to the replication of 21 element vector given in the top part of Figure4c with the period of  $10f_s = 20$  samples. This causes an aliasing in the time-domain, but as can be seen from Figure4d, time-domain aliasing causes a change in the values of two samples at the very edges of interval  $[-5, 5]$ . As can be seen from Figure4d, these samples (which are the two lowest valued samples in Figure4d,) have almost no change due to time aliasing.

**Case 4 (no aliasing):** We sample the time-domain signal with  $f_s = 2$  Hz as before. Again, we have a discrete time signal with 21 non-zero entries (approximately!) after time-domain sampling. Different from the earlier case, we evaluate the spectrum at  $N_{\text{point}} = 30$  points as shown in Figure5c. This leads to no aliasing in the time-domain. Furthermore, as can be seen from Figure5d, the periodic extension with 30 samples, results in an insertion of 9 zeros in the middle portion of the time-domain representation shown in Figure5d. The insertion of additional zeros is called as zero-padding in the DFT literature.

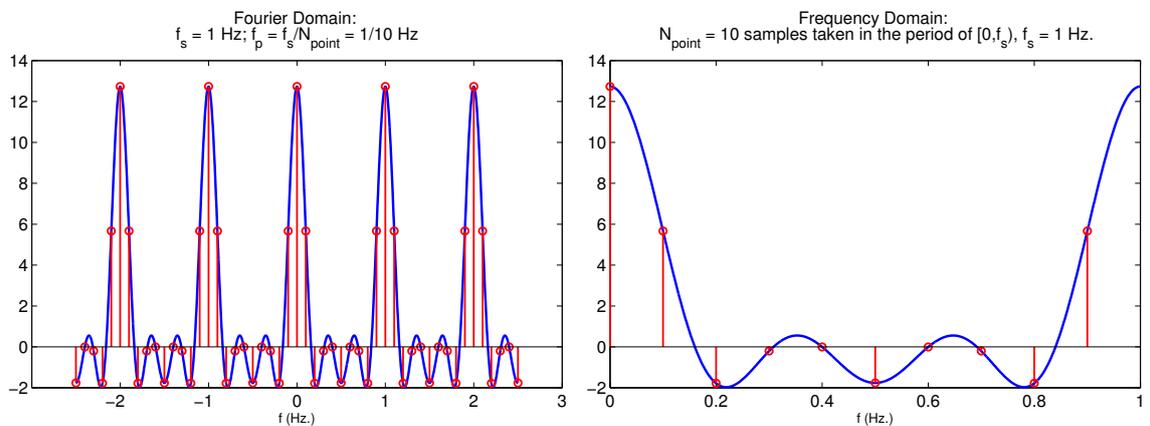
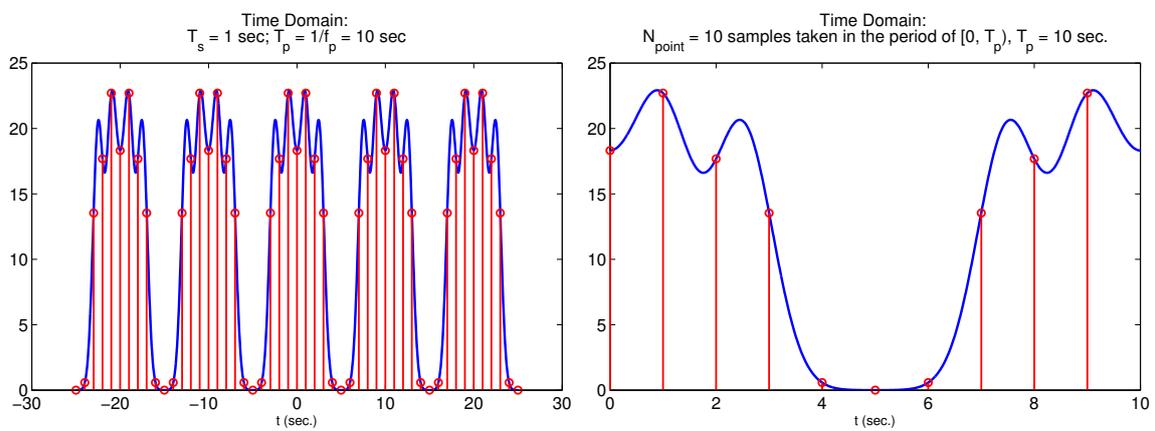
We conclude our findings with the following fact statement.

**Fact 4 (Calculation of Spectrum Samples by DFT)** *In order to accurately calculate the spectrum samples of a continuous function, 1) the function should be sampled at a sufficiently high rate; 2) Sufficiently high number of time-domain samples should be collected so that the practical time-support of the function is covered; 3)  $N_{\text{point}}$  point DFT of the samples should be calculated where  $N_{\text{point}}$  can be as large as number of collected samples or higher.*



(a) Step 0: Time and Frequency Domain

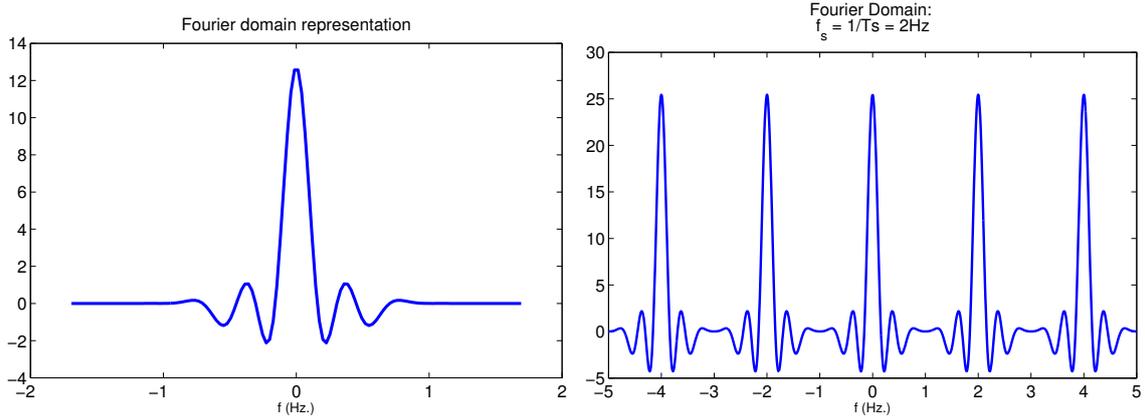
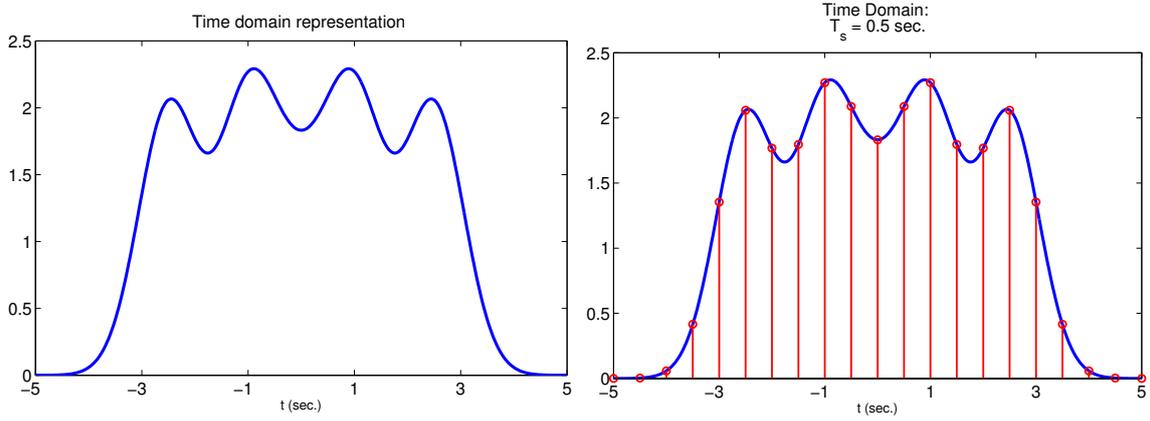
(b) Step 1: Time Sampling



(c) Step 2: Frequency Sampling

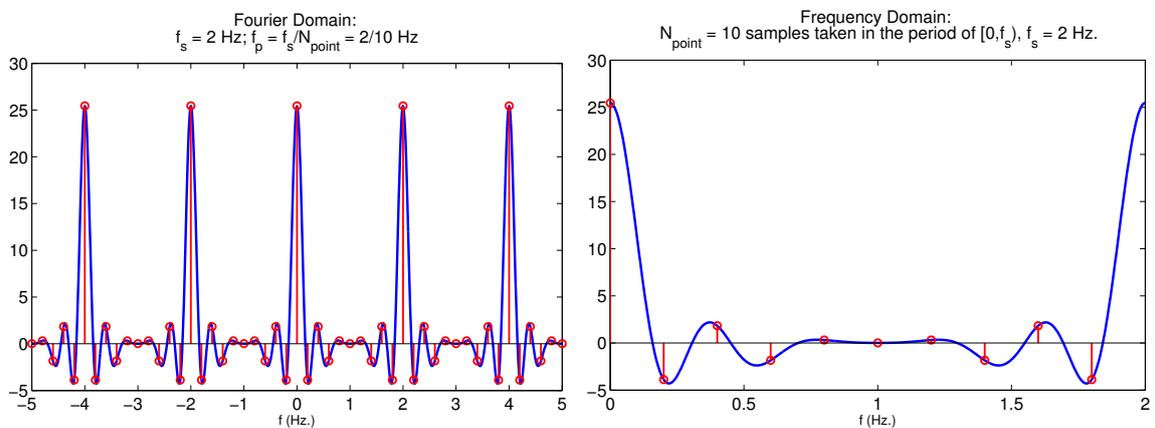
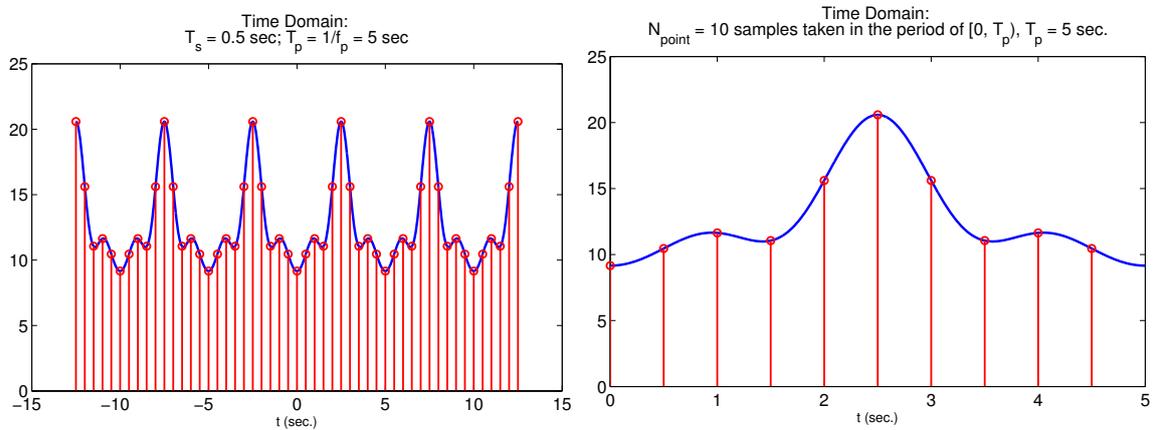
(d) Step 3:  $N_{\text{point}} = 10$  samples in both domains

Figure 2:  $N_{\text{point}} = 10$  time and frequency domain samples,  $f_s = 1, T_p = 10$ . (The case of evident frequency aliasing)



(a) Step 0: Time and Frequency Domain

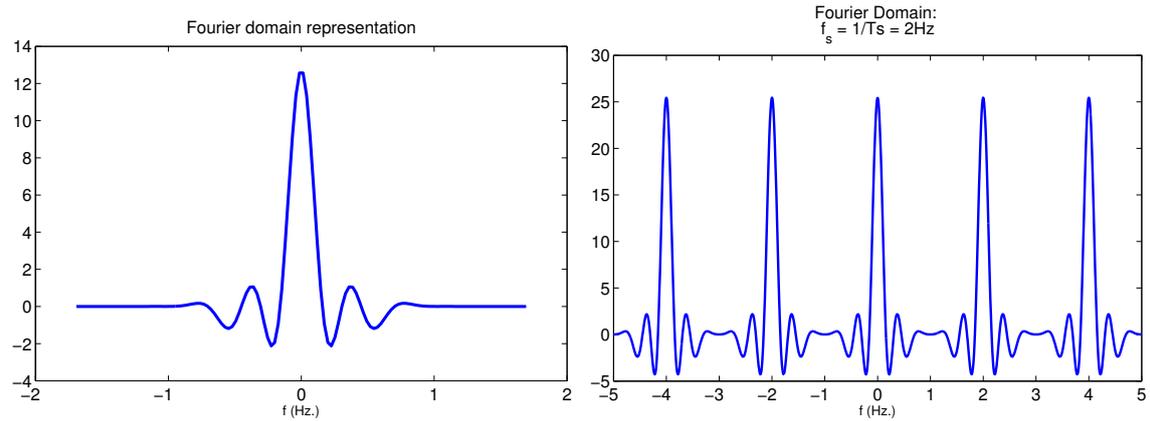
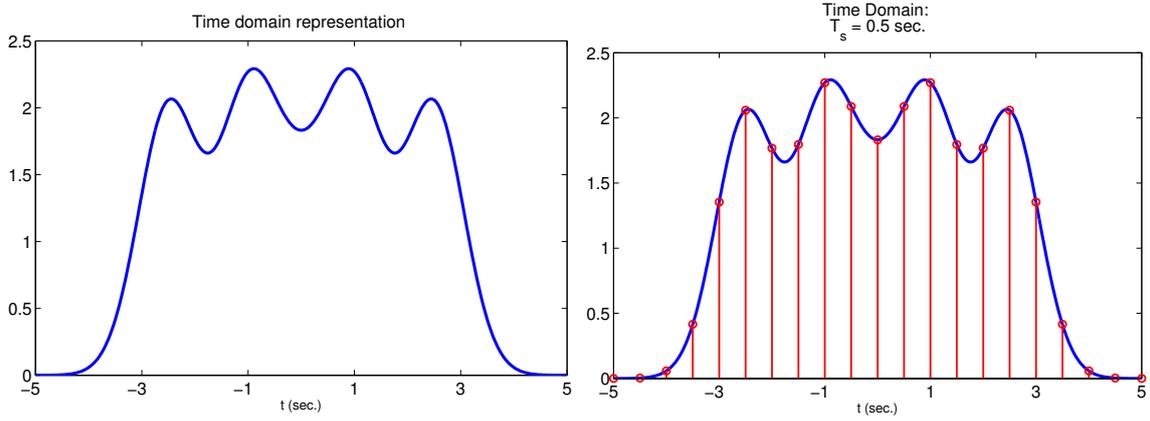
(b) Step 1: Time Sampling



(c) Step 2: Frequency Sampling

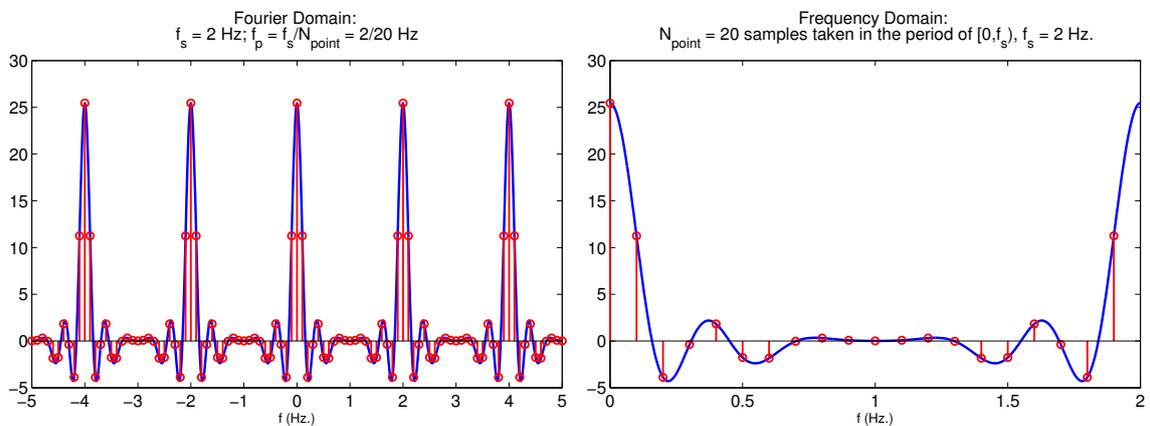
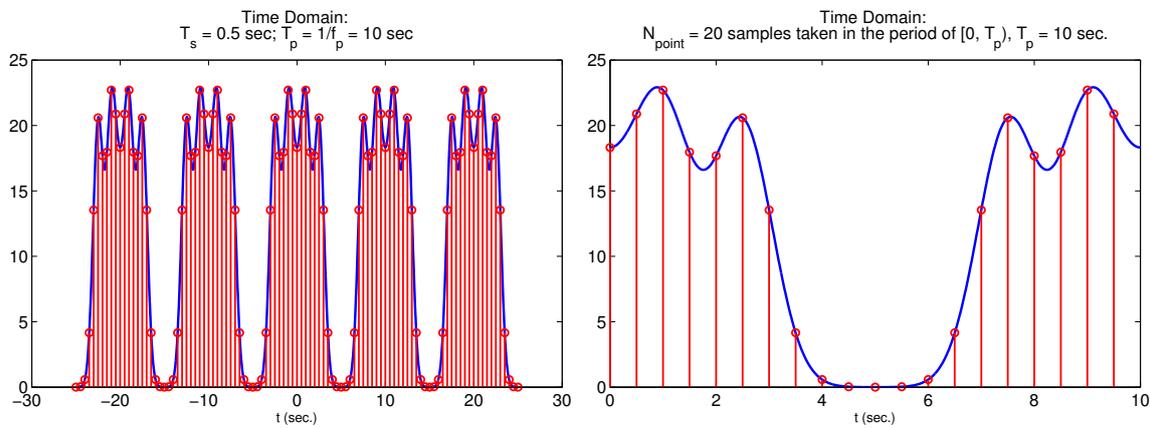
(d) Step 3:  $N_{\text{point}} = 10$  samples in both domains

Figure 3:  $N_{\text{point}} = 10$  time and frequency domain samples,  $f_s = 2, T_p = 5$ . (The case of evident time aliasing)



(a) Step 0: Time and Frequency Domain

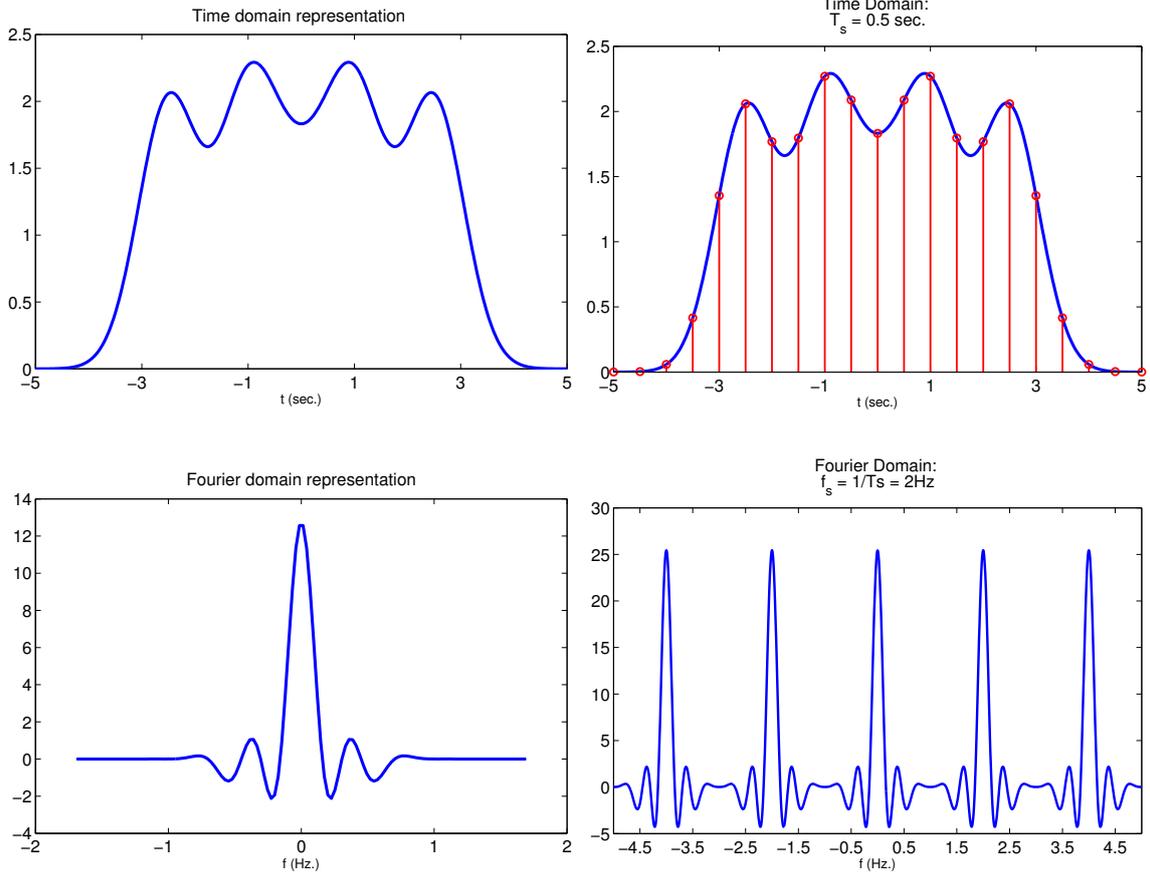
(b) Step 1: Time Sampling



(c) Step 2: Frequency Sampling

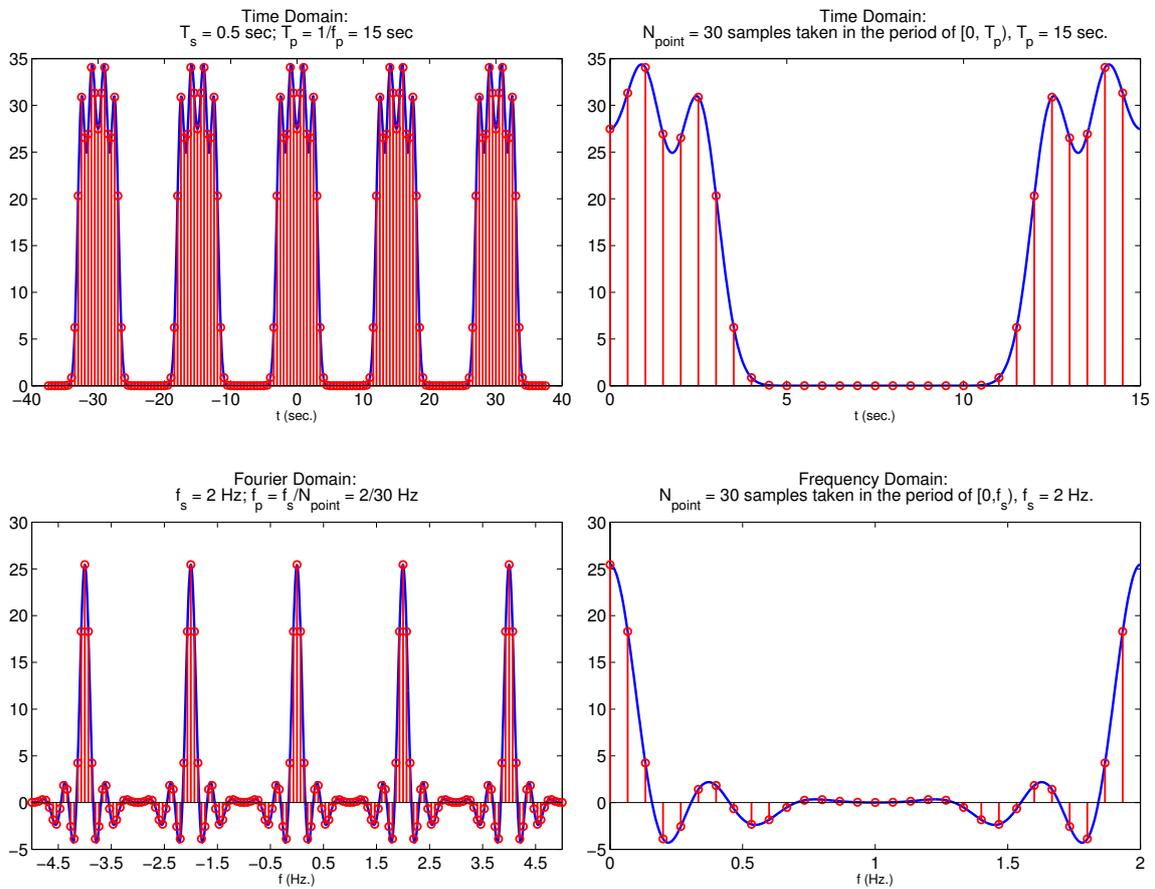
(d) Step 3:  $N_{\text{point}} = 20$  samples in both domains

Figure 4:  $N_{\text{point}} = 20$  time and frequency domain samples,  $f_s = 2, T_p = 10$ . (No evident aliasing in both domains)



(a) Step 0: Time and Frequency Domain

(b) Step 1: Time Sampling



(c) Step 2: Frequency Sampling

(d) Step 3:  $N_{\text{point}} = 30$  samples in both domains

Figure 5:  $N_{\text{point}} = 30$  time and frequency domain samples,  $f_s = 2$ ,  $T_p = 15$ . (No aliasing)

### 3 Poisson Summation Formula

As shown in (17), the  $\tilde{f}_{T_p}[n]$  and  $T_p\tilde{F}_{f_s}[k]$  are related with  $N_{\text{point}}$  DFT operation. Remembering that the value at the DC bin of the DFT output is identical to the cumulative sum of the input, we can write the following for the cumulative sum of input:

$$\begin{aligned}
\sum_{n=0}^{N_{\text{point}}-1} \tilde{f}_{T_p}[n] &= T_p\tilde{F}_{f_s}[0] \\
&\stackrel{(14)}{=} T_p\tilde{F}_{f_s}(f_pk) \downarrow_{k=0} \\
&\stackrel{(12)}{=} T_p f_s \sum_{k'=-\infty}^{\infty} F(f_pk - k'f_s) \downarrow_{k=0} \\
&= T_p f_s \sum_{k=-\infty}^{\infty} F(kf_s). \tag{18}
\end{aligned}$$

The cumulative sum appearing on the left hand side of the equation (18) can also written as

$$\begin{aligned}
\sum_{n=0}^{N_{\text{point}}-1} \tilde{f}_{T_p}[n] &\stackrel{(15)}{=} T_p \sum_{n=0}^{N_{\text{point}}-1} \sum_{k=-\infty}^{\infty} f[n - kN_{\text{point}}] \\
&\stackrel{(11)}{=} T_p \sum_{n=0}^{N_{\text{point}}-1} \sum_{k=-\infty}^{\infty} f((n - kN_{\text{point}})T_s) \\
&= T_p \sum_{n=-\infty}^{\infty} f(nT_s). \tag{19}
\end{aligned}$$

Equating the right hand sides of (18) and (19), we immediately get the general form of Poisson summation formula,

$$\sum_{k=-\infty}^{\infty} F(kf_s) = T_s \sum_{n=-\infty}^{\infty} f(nT_s). \tag{20}$$

The most well known case for this relation, is the special case of  $T_s = 1$  for which the formula reduces to

$$\sum_{k=-\infty}^{\infty} F(k) = \sum_{n=-\infty}^{\infty} f(n) \tag{21}$$

where  $f(t)$  and  $F(f)$  are the Fourier transform pairs given by the definitions in (1).

**Example:** The  $f(t) = \exp(-|t|)$  and  $F(f) = \frac{2}{1+(2\pi f)^2}$  are Fourier transform pairs. The application of Poisson summation formula leads to the equality of the following sums:

$$\sum_{n=-\infty}^{\infty} f(n) = 2 \sum_{n=1}^{\infty} e^{-n} + 1 = 2 \frac{e^{-1}}{1 - e^{-1}} + 1 = \frac{e + 1}{e - 1}, \tag{22}$$

$$\sum_{k=-\infty}^{\infty} F(k) = \sum_{k=-\infty}^{\infty} \frac{2}{1 + 4\pi^2 k^2}. \tag{23}$$

The final equality can be written as

$$\sum_{k=-\infty}^{\infty} \frac{1}{1 + 4\pi^2 k^2} = \frac{e + 1}{2(e - 1)}. \tag{24}$$

Repeating the same calculation with  $T_s = 2\pi$  with the general relation given in (20), we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{1 + k^2} = \pi \frac{e^{2\pi} + 1}{e^{2\pi} - 1} \tag{25}$$

which is a surprising relation. Several other equally surprising relations can be retrieved similarly.

## 4 Appendices

### 4.1 Matlab Codes For The Generation of Figures in Section 2

```
1 %Sampling Rates
2 fs = 2; Npoint = 20;
3
4 %Function Declarations
5 a = 1/2; sigma = sqrt(1/2/a/pi);
6 gt = @(t) exp(-pi*a*t.^2);
7 Gf = @(f) 1/sqrt(a)*exp(-pi/a*f.^2);
8 myfunct = @(t) 1*gt(t) + 2*gt(t-1) + 2*gt(t+1) + 2*gt(t-2.5) + 2*gt(t+2.5);
9 myfuncf = @(f) 1*Gf(f) + 2*Gf(f).*exp(-j*2*pi*f*1) + 2*Gf(f).*exp(j*2*pi*f*1)...
10         + 2*Gf(f).*exp(-j*2*pi*f*2.5) + 2*Gf(f).*exp(+j*2*pi*f*2.5);
11
12 %%%%%%%%%%
13 %Figure 1
14 %%%%%%%%%%
15 t = linspace(-5,5,512);
16 f = linspace(-3*sigma,3*sigma,128);
17
18 figure(1),
19 subplot(211),set(gca,'fontsize',12)
20 plot(t,myfunct(t),'linewidth',2.5);
21 title('Time domain representation ');
22 xlabel('t (sec.)','fontsize',11);
23 set(gca,'xtick',-5:2:5)
24
25 subplot(212),set(gca,'fontsize',12);
26 plot(f,real(myfuncf(f)),'linewidth',2.5);
27 title('Fourier domain representation');
28 xlabel('f (Hz.)','fontsize',11);
29 set(gca,'xtick',-2:1:2)
30
31 %Sanity check:
32 % area_under_gt = quad(myfunct,-5,5);
33 % dc_value_gf = myfuncf(0);
34 % [area_under_gt dc_value_gf]
35
36 %%%%%%%%%%
37 %Figure 2
38 %%%%%%%%%%
39 Ts=1/fs; %Sampling period
40 tsamples = -5:Ts:5; %sampling times
41 myfuncn = myfunct(tsamples); %samples
42 numsamples = length(myfuncn); %number of samples
43
44 figure(2),
45 subplot(211),set(gca,'fontsize',12)
46 plot(t,myfuncn(t),'linewidth',2.5); hold on;
47 stem(tsamples,myfuncn,'or','linewidth',1.5); hold off;
48 title(['Time Domain: ' char(10) 'T_s = ' num2str(Ts) ' sec.']);
49 xlabel('t (sec.)','fontsize',11);
50 set(gca,'xtick',-5:2:5)
51
52 subplot(212), set(gca,'fontsize',12);
53 fx = linspace(-fs*2.5,fs*2.5,1024); %frequency axis for the periodic func.
54 myfuncfp = @(fx) fs*(myfuncf(fx) + myfuncf(fx-fs) + myfuncf(fx+fs) ...
55                 + myfuncf(fx -2*fs) + myfuncf(fx+2*fs) ...
56                 + myfuncf(fx -3*fs) + myfuncf(fx+3*fs) ...
57                 + myfuncf(fx -4*fs) + myfuncf(fx+4*fs));
58                 %periodic signal in frequency due to time sampling
59 plot(fx,myfuncfp(fx),'linewidth', 2);
60 title(['Fourier Domain: ' char(10) 'f_s = 1/Ts = ' num2str(fs) ' Hz']);
61 xlabel('f (Hz.)','fontsize',11);
```

```

62 set(gca,'xtick',(-fs*2.5:1:fs*2.5) + 0.5)
63
64
65 %%%%%%%%%%
66 %Figure 3
67 %%%%%%%%%%
68 fp = fs/Npoint; %Frequency Sampling (Npoint points are taken in
69 % a full period of periodic function in frequency)
70 Tp = 1/fp;
71 tx = linspace(-Tp*2.5,Tp*2.5,1024);
72 %tx: axis for the corresponding time-periodic function
73 % due to frequency sampling
74 txsamples= 0:Ts:Tp*2.5; txsamples = [-txsamples(end:-1:2) txsamples];
75 myfunctp = @(tx) Tp*(myfunct(tx) + myfunct(tx-Tp) ...
76 + myfunct(tx+Tp) ...
77 + myfunct(tx -2*Tp) + myfunct(tx+2*Tp) ...
78 + myfunct(tx -3*Tp) + myfunct(tx+3*Tp) ...
79 + myfunct(tx -4*Tp) + myfunct(tx+4*Tp));
80
81 figure(3),
82 subplot(211),set(gca,'fontsize',12)
83 plot(tx,myfunctp(tx),'linewidth', 2); hold on,
84 stem(txsamples,myfunctp(txsamples),'or','linewidth',1.5); hold off;
85 title(['Time Domain: ' char(10) 'T_s = ' num2str(Ts) ...
86 ' sec; T_p = 1/f_p = ' num2str(Tp) ' sec']);
87 xlabel('t (sec.)','fontsize',11);
88
89
90 subplot(212), set(gca,'fontsize',12);
91 fxsamples = 0:fp:fs*2.5; fxsamples = [-fxsamples(end:-1:2) fxsamples];
92 plot(fx,myfuncfp(fx),'linewidth', 2); hold on;
93 stem(fxsamples,myfuncfp(fxsamples),'or','linewidth', 1.5);
94 hold off;
95 title(['Fourier Domain: ' char(10) 'f_s = ' num2str(fs) ...
96 ' Hz;', ' f_p = f_s/N_{point} = ' num2str(fs) '/' num2str(Npoint) ...
97 ' Hz']);
98 xlabel('f (Hz.)','fontsize',11);
99 set(gca,'xtick',(-fs*2.5:1:fs*2.5) + 0.5)
100
101
102 %%%%%%%%%%
103 %Figure 4
104 %%%%%%%%%%
105 figure(4),
106 subplot(211),set(gca,'fontsize',12)
107 tx_p = linspace(0,Tp,1024); %The interval for the single period
108 % of the time-periodic function
109 txsamples_p= 0:Ts:(Tp - Ts); %Sampling points in a single period of
110 % the time-periodic function
111 % in [0,Tp).
112 plot(tx_p,myfunctp(tx_p),'linewidth', 2); hold on,
113 DFT_input = myfunctp(txsamples_p);
114 stem(txsamples_p,DFT_input,'or','linewidth',1.5); hold off;
115 title(['Time Domain: ' char(10) 'N_{point} = ' num2str(Npoint) ...
116 ' samples taken in the period of [0, T_p), T_p = ' num2str(Tp) ...
117 ' sec.']);
118 xlabel('t (sec.)','fontsize',11);
119
120
121 subplot(212), set(gca,'fontsize',12);
122 fx_p = linspace(0,fs,1024); %The interval for the single period
123 % of the frequency-periodic function
124 fxsamples_p = 0:fp:(fs-fp); %Sampling points in a single period of
125 % the frequency-periodic function
126 % in [0,fs).
127 plot(fx_p,myfuncfp(fx_p),'linewidth', 2); hold on;
128 DFT_output = myfuncfp(fxsamples_p);

```

```
129 stem(fxsamples_p,DFT_output,'or','linewidth',1.5);
130 title(['Frequency Domain: ' char(10) 'N-{'point} = ' num2str(Npoint) ...
131       ' samples taken in the period of [0,f_s), f_s = ' num2str(fs)...
132       ' Hz.']);
133 hold off;
134 xlabel('f (Hz.)','fontsize',11);
135
136
137 %End of Figures
138
139 disp('Are samples in a period related through DFT operation or not?');
140 [fs/Npoint*real(fft(DFT_input)); DFT_output]
```